# Tutorial 9: Errata to sup-norm method

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23/3/2018

In tutorial, I said the following "sup-norm method" of determining non-uniform convergence of sequence of functions, as follows:

"Proposition" Given a sequence of functions $\{f_n : I \to \mathbb{R}\}$

if $\lim_{n\to\infty} \|f_n\| = +\infty$, then $\{f_n\}$ does not converge uniformly.

However, the above proposition is false in general: consider $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = x + \frac{1}{n}$

then $\|f_n\| = +\infty$ for all $n \in \mathbb{N}$, hence $\lim_{n\to\infty} \|f_n\| = +\infty$

but define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x$

then $\{f_n\}$ converges uniformly to $f$ on $\mathbb{R}$,

as $\|f_n - f\| = \frac{1}{n} \to 0$ as $n \to \infty$.

The corrected version of "sup-norm method" is as follows:

Proposition Given a sequence of functions $\{f_n : I \to \mathbb{R}\}$

if $\lim_{n\to\infty} \|f_n\| = +\infty$, and $f_n$ converges

pointwisely to a bounded function $f: I \to \mathbb{R}$

then $\{f_n\}$ does not converge uniformly.

Proof: Suppose $\{f_n\}$ converges uniformly, then

by uniqueness of limit, $\{f_n\}$ converges to $f$ uniformly.

Choose $\varepsilon = 1$; by Proposition 3.1,

there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\|f_n - f\| \leq 1$$

Therefore, for all $n \geq N$, $\forall x \in I$,

$$|f_n(x)| \leq |f_n(x) - f(x)| + |f(x)| \leq \|f_n - f\| + \|f\| \leq 1 + \|f\|$$

$\therefore \|f_n\| \leq 1 + \|f\|$, $\forall n \geq N$. contradicting $\lim_{n\to\infty} \|f_n\| = +\infty$

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e.g. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \frac{1}{n} x$

then $\|f_n\| = +\infty$, $\therefore \lim_{n\to\infty} \|f_n\| = \infty$

pointwise limit is clearly $f(x) \equiv 0$, and hence is bounded.

Therefore, by proposition, $\{f_n\}$ is not uniformly convergent.